Raíz cúbica de un número complejo

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Sea z un número complejo en el círculo unitario.

$$z = \cos(\alpha) + i\sin(\alpha)$$

deseamos encontrar la raíz cúbica de z, es decir, deseamos encontrar r tal que: $r^3 = z$. Por la fórmula de De Moivre sabemos que:

$$r = \cos\left(\frac{\alpha}{3}\right) + i\sin\left(\frac{\alpha}{3}\right)$$

Entonces, resolver este problema se reduce a encontrar $\cos\left(\frac{\alpha}{3}\right)$ y $\sin\left(\frac{\alpha}{3}\right)$ en términos de $\cos(\alpha)$ y $\sin(\alpha)$.

Consideremos entonces:

$$\begin{aligned}
e^{3\varphi i} &= \left(e^{\varphi i}\right)^3 \\
&= (\cos(\varphi) + i\sin(\varphi))^3 \\
&= \cos(\varphi)^3 + 3i\cos(\varphi)^2 sin(\varphi) - 3\cos(\varphi) sin(\varphi)^2 - i\sin(\varphi)^3 \\
&= \cos(\varphi)^3 - 3\cos(\varphi) sin(\varphi)^2 + 3i\cos(\varphi)^2 sin(\varphi) - i\sin(\varphi)^3 \\
&= \cos(\varphi)^3 - 3\cos(\varphi)\left(1 - cos(\varphi)^2\right) + 3i\left(1 - \sin(\varphi)^2\right)\sin(\varphi) - i\sin(\varphi)^3 \\
&= \cos(\varphi)^3 - 3\cos(\varphi) + 3cos(\varphi)^3 + 3i\sin(\varphi) - 3i\sin(\varphi)^3 - i\sin(\varphi)^3 \\
&=4\cos(\varphi)^3 - 3\cos(\varphi) + i\left(3\sin(\varphi) - 4\sin(\varphi)^3\right)
\end{aligned}$$

es decir:

$$\begin{aligned}
\cos(3\varphi) &=4\cos(\varphi)^3 - 3\cos(\varphi) \\
\sin(3\varphi) &=3\sin(\varphi) - 4\sin(\varphi)^3 .
\end{aligned}$$

Si tomamos $\varphi = \frac{\alpha}{3}$

$$\begin{aligned}
\cos(\alpha) &=4\cos\left(\frac{\alpha}{3}\right)^3 - 3\cos\left(\frac{\alpha}{3}\right) \\
\sin(\alpha) &=3\sin\left(\frac{\alpha}{3}\right) - 4\sin\left(\frac{\alpha}{3}\right)^3 .
\end{aligned}$$

que podemos reacomodar:

$$
\begin{aligned}
4\cos\left(\frac{\alpha}{3}\right)^3 - 3\cos\left(\frac{\alpha}{3}\right) - \cos(\alpha) = 0\\
4\sin\left(\frac{\alpha}{3}\right)^3 - 3\sin\left(\frac{\alpha}{3}\right) + \sin(\alpha) = 0.
\end{aligned}
$$

o bien:

$$\cos\left(\frac{\alpha}{3}\right)^3 - \frac{3}{4}\cos\left(\frac{\alpha}{3}\right) - \frac{1}{4}\cos(\alpha) = 0 \tag{1}$$

$$\sin\left(\frac{\alpha}{3}\right)^3 - \frac{3}{4}\sin\left(\frac{\alpha}{3}\right) + \frac{1}{4}\sin(\alpha) = 0. \tag{2}$$

que son ecuaciones cúbicas de la forma:

$$t^3 + px + q = 0$$

cuyas soluciones pueden darse como sigue. Sean:

$$
\begin{aligned}
d =& \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\\
u =& \sqrt[3]{-\frac{q}{2} + d}\\
v =& \sqrt[3]{-\frac{q}{2} - d}
\end{aligned}
$$

y, suponiendo que se cumple:

$$uv = -\frac{p}{3}$$

entonces, las soluciones son:

$$
\begin{aligned}
t_1 =& u + v\\
t_2 =& u\omega^2 + v\omega\\
t_3 =& u\omega + v\omega^2
\end{aligned}
$$

donde

$$
\begin{aligned}
\omega =& -\frac{1}{2} + \frac{\sqrt{3}}{2}i\\
\omega^2 =& -\frac{1}{2} - \frac{\sqrt{3}}{2}i
\end{aligned}
$$

Empecemos por resolver (1). Tenemos:

$$
\begin{aligned}
p_c =& -\frac{3}{4}\\
q_c =& -\frac{1}{4}\cos(\alpha)
\end{aligned}
$$

Sustituyendo, tenemos

$$\begin{aligned}
d_c =& \sqrt{\left(\frac{q_c}{2}\right)^2 + \left(\frac{p_c}{3}\right)^3} \\
=& \sqrt{\left(\frac{-\frac{1}{4}\cos(\alpha)}{2}\right)^2 + \left(\frac{-\frac{3}{4}}{3}\right)^3} \\
=& \sqrt{\left(\frac{\cos(\alpha)}{8}\right)^2 + \left(-\frac{1}{4}\right)^3} \\
=& \sqrt{\frac{\cos\left(\alpha\right)^2}{64} - \frac{1}{64}} \\
=& \frac{1}{8}\sqrt{\cos\left(\alpha\right)^2 - 1} \\
=& \pm\frac{i}{8}\sin\left(\alpha\right)
\end{aligned}$$

de donde:

$$\begin{aligned}
u_c =& = \sqrt[3]{-\frac{q_c}{2} + d_c} \\
=& \sqrt[3]{-\frac{\left(-\frac{\cos(\alpha)}{4}\right)}{2} + \frac{i}{8}\sin\left(\alpha\right)} \\
=& \sqrt[3]{\frac{\cos\left(\alpha\right)}{8} + \frac{i}{8}\sin\left(\alpha\right)} \\
=& \frac{1}{2}\sqrt[3]{\cos\left(\alpha\right) + i\sin\left(\alpha\right)} \\
u_{ck} =& \frac{1}{2}e^{\frac{2k\pi+\alpha}{3}i}, \quad k = 0, 1, 2
\end{aligned}$$

$$\begin{aligned}
v_c =& = \sqrt[3]{-\frac{q_c}{2} - d_c} \\
=& \sqrt[3]{-\frac{\left(-\frac{\cos(\alpha)}{4}\right)}{2} - \frac{i}{8}\sin\left(\alpha\right)} \\
=& \sqrt[3]{\frac{\cos\left(\alpha\right)}{8} - \frac{i}{8}\sin\left(\alpha\right)} \\
=& \frac{1}{2}\sqrt[3]{\cos\left(\alpha\right) - i\sin\left(\alpha\right)} \\
=& \frac{1}{2}\sqrt[3]{\cos\left(-\alpha\right) + i\sin\left(-\alpha\right)} \\
v_{ck} =& \frac{1}{2}e^{\frac{2k\pi-\alpha}{3}i}, \quad k = 0, 1, 2
\end{aligned}$$

Debemos recordar que la solución debe satisfacer:

$$uv = -\frac{p}{3} = -\frac{\left(-\frac{3}{4}\right)}{3} = \frac{1}{4}$$

Las parejas que cumplen con la condición anterior son:

$$\begin{aligned}
u_{c0}v_{c0} &= \left(\frac{1}{2}e^{\frac{\alpha}{3}i}\right)\left(\frac{1}{2}e^{\frac{-\alpha}{3}i}\right) = \frac{1}{4}\\
u_{c1}v_{c2} &= \left(\frac{1}{2}e^{\left(\frac{2\pi+\alpha}{3}\right)i}\right)\left(\frac{1}{2}e^{\left(\frac{4\pi-\alpha}{3}\right)i}\right) = \frac{1}{4}e^{2\pi i} = \frac{1}{4}\\
u_{c2}v_{c1} &= \left(\frac{1}{2}e^{\left(\frac{4\pi+\alpha}{3}\right)i}\right)\left(\frac{1}{2}e^{\left(\frac{2\pi-\alpha}{3}\right)i}\right) = \frac{1}{4}e^{2\pi i} = \frac{1}{4}
\end{aligned}$$

de manera que las soluciones de la ecuación son:

$$t_{c0} = u_{c0}v_{c0} = \frac{1}{2}\left(e^{\frac{\alpha}{3}i} + e^{-\frac{\alpha}{3}i}\right)$$

que, irónicamente:

$$\begin{aligned}
&= \cos\left(\frac{\alpha}{3}\right).\\
t_{c1} &= u_{c1}v_{c2} = \frac{1}{2}\left(e^{\frac{2\pi+\alpha}{3}i} + e^{\frac{4\pi-\alpha}{3}i}\right)\\
&= \frac{1}{2}\left(e^{\frac{2\pi}{3}i}e^{\frac{\alpha}{3}i} + e^{\frac{4\pi}{3}i}e^{-\frac{\alpha}{3}i}\right)\\
&= \frac{1}{2}\left(e^{\frac{2\pi}{3}i}e^{\frac{\alpha}{3}i} + e^{-\frac{2\pi}{3}i}e^{-\frac{\alpha}{3}i}\right)\\
&= \frac{1}{2}\left(e^{\frac{2\pi+\alpha}{3}i} + e^{-\frac{2\pi-\alpha}{3}i}\right)\\
&= \cos\left(\frac{2\pi+\alpha}{3}\right)\\
t_{c2} &= u_{c2}v_{c1} = \frac{1}{2}\left(e^{\frac{4\pi+\alpha}{3}i} + e^{\frac{2\pi-\alpha}{3}i}\right)\\
&= \frac{1}{2}\left(e^{\frac{4\pi}{3}i}e^{\frac{\alpha}{3}i} + e^{\frac{2\pi}{3}i}e^{-\frac{\alpha}{3}i}\right)\\
&= \frac{1}{2}\left(e^{\frac{4\pi}{3}i}e^{\frac{\alpha}{3}i} + e^{-\frac{4\pi}{3}i}e^{-\frac{\alpha}{3}i}\right)\\
&= \frac{1}{2}\left(e^{\frac{4\pi+\alpha}{3}i} + e^{-\frac{4\pi-\alpha}{3}i}\right)\\
&= \cos\left(\frac{4\pi+\alpha}{3}\right)
\end{aligned}$$

Ahora resolveremos (2).

$$\sin\left(\frac{\alpha}{3}\right)^3 - \frac{3}{4}\sin\left(\frac{\alpha}{3}\right) + \frac{1}{4}\sin(\alpha) = 0.$$

La ecuación tiene la forma:

$$t^3 - \frac{3}{4}t + \frac{1}{4}\sin(\alpha) = 0.$$

Para esta ecuación:

$$\begin{aligned} p_s =& -\frac{3}{4} \\ q_s =& \frac{1}{4}\sin(\alpha). \end{aligned}$$

Tenemos:

$$\begin{aligned} d_s =& \sqrt{\left(\frac{q_s}{2}\right)^2 + \left(\frac{p_s}{3}\right)^3} \\ =& \sqrt{\left(\frac{\frac{1}{4}\sin(\alpha)}{2}\right)^2 + \left(\frac{-\frac{3}{4}}{3}\right)^3} \\ =& \sqrt{\left(\frac{\sin(\alpha)}{8}\right)^2 - \left(\frac{1}{4}\right)^3} \\ =& \sqrt{\frac{\sin(\alpha)^2}{64} - \frac{1}{64}} \\ =& \frac{1}{8}\sqrt{\sin(\alpha)^2 - 1} \\ =& \pm\frac{i}{8}\cos(\alpha) \end{aligned}$$

$$\begin{aligned} u_s =& = \sqrt[3]{-\frac{q_s}{2} + d_s} \\ =& \sqrt[3]{-\frac{\left(\frac{\sin(\alpha)}{4}\right)}{2} + \frac{i}{8}\cos(\alpha)} \\ =& \sqrt[3]{-\frac{\sin(\alpha)}{8} + \frac{i}{8}\cos(\alpha)} \\ =& \frac{1}{2}\sqrt[3]{-\sin(\alpha) + i\cos(\alpha)} \\ =& \frac{1}{2}\sqrt[3]{\cos\left(\frac{\pi}{2}+\alpha\right) + i\sin\left(\frac{\pi}{2}+\alpha\right)} \\ u_{sk} =& \frac{1}{2}e^{\left(\frac{2k\pi}{3} + \frac{\frac{\pi}{2}+\alpha}{3}\right)i}, \quad k = 0, 1, 2 \end{aligned}$$

$$
\begin{aligned}
v_s = &= \sqrt[3]{-\frac{q_s}{2} - d_s} \\
&= \sqrt[3]{-\frac{\left(\frac{\sin(\alpha)}{4}\right)}{2} - \frac{i}{8}\cos(\alpha)} \\
&= \sqrt[3]{-\frac{\sin(\alpha)}{8} - \frac{i}{8}\cos(\alpha)} \\
&= -\frac{1}{2}\sqrt[3]{\sin(\alpha) + i\cos(\alpha)} \\
&= -\frac{1}{2}\sqrt[3]{\cos\left(\frac{\pi}{2} - \alpha\right) + i\sin\left(\frac{\pi}{2} - \alpha\right)} \\
v_{sk} &= \frac{1}{2}e^{\left(\frac{2k\pi}{3} + \frac{\frac{\pi}{2} - \alpha}{3}\right)i}, \quad k = 0, 1, 2
\end{aligned}
$$

Las parejas (u, v) deben satisfacer:

$$uv = -\frac{p}{3} = -\frac{-\frac{3}{4}}{3} = \frac{1}{4}$$

Las que satisfacen son:

$$u_{s0}v_{s0} = \frac{1}{2}e^{\left(\frac{2k\pi}{3} + \frac{\frac{\pi}{2} + \alpha}{3}\right)i}\frac{1}{2}e^{\left(\frac{2k\pi}{3} + \frac{\frac{\pi}{2} - \alpha}{3}\right)i}$$

Lo cual es una soberana locura, porque no queríamos usar la fórmula de De Moivre, pero hemos llegado a ella...