$x = 2\cos\left(\frac{2\pi}{15}\right)$, satisface $x^4 - x^3 - 4x^2 + 4x + 1 = 0$

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La estrategia va a ser un poco rebuscada:

1. Calcularemos primero $\cos\left(\frac{2\pi}{5}\right)$ y $\sin\left(\frac{2\pi}{5}\right)$
2. Dado que conocemos $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ y $\sin\left(\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right)$ y como $\frac{2\pi}{5} - \frac{\pi}{3} = \frac{\pi}{15}$, entonces calcularemos $\cos\left(\frac{\pi}{15}\right)$.
3. Como sabemos que $cos(2\alpha) = 2cos(\alpha)^2 - 1$, podremos calcular $\cos\left(\frac{2\pi}{15}\right)$
4. Sustituiremos $x = 2\cos\left(\frac{2\pi}{15}\right)$ en la ecuación propuesta y veremos si se satisface.

1. Cálculo de $\cos\left(\frac{2\pi}{5}\right)$ y $\sin\left(\frac{2\pi}{5}\right)$

Partimos de tres conocidas identidades:

$$1 = \cos(\alpha)^2 + \sin(\alpha)^2 \tag{1}$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{2}$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{3}$$

De estas identidades, para $\beta = \alpha$:

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{4}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{5}$$

Consideremos (5) (4):

$$1 = \cos^2(\alpha) + \sin^2(\alpha)$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

Sumando y restando obtenemos, respectivamente:

$$1 + \cos(2\alpha) = 2\cos^2(\alpha)$$

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1 \tag{6}$$

$$1 - \cos(2\alpha) = 2\sin^2(\alpha)$$

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha) \tag{7}$$

Calculamos $\cos(3\alpha)$ usando (2) con $\beta = 2\alpha$:

$$\begin{aligned}
\cos(3\alpha) =& \cos(\alpha)\cos(2\alpha) - \sin(\alpha)\sin(2\alpha) \\
=& \cos(\alpha)(2\cos^2(\alpha) - 1) - \sin(\alpha)(2\sin(\alpha)\cos(\alpha)) \\
=& 2\cos^3(\alpha) - \cos(\alpha) - 2\sin^2(\alpha)\cos(\alpha) \\
=& 2\cos^3(\alpha) - \cos(\alpha) - 2(1 - \cos^2(\alpha))\cos(\alpha) \\
=& 2\cos^3(\alpha) - \cos(\alpha) - 2\cos(\alpha) + 2\cos^3(\alpha)) \\
\cos(3\alpha) =& 4\cos^3(\alpha) - 3\cos(\alpha) \qquad (8)
\end{aligned}$$

Calculamos $\sin(3\alpha)$ usando (3) con $\beta = 2\alpha$:

$$\begin{aligned}
\sin(3\alpha) =& \sin(\alpha)\cos(2\alpha) + \cos(\alpha)\sin(2\alpha) \\
=& \sin(\alpha)(1 - 2\sin^2(\alpha)) + \cos(\alpha)(2\sin(\alpha)\cos(\alpha)) \\
=& \sin(\alpha) - 2\sin^3(\alpha) + 2\sin(\alpha)\cos^2(\alpha) \\
=& \sin(\alpha) - 2\sin^3(\alpha) + 2\sin(\alpha)(1 - \sin^2(\alpha)) \\
=& \sin(\alpha) - 2\sin^3(\alpha) + 2\sin(\alpha) - 2\sin^3(\alpha)) \\
\sin(3\alpha) =& 3\sin(\alpha) - 4\sin^3(\alpha) \qquad (9)
\end{aligned}$$

Calculamos $\cos(5\alpha)$:

$$\begin{aligned}
\cos(5\alpha) =& \cos(3\alpha)\cos(2\alpha) - \sin(3\alpha)\sin(2\alpha) \\
=& (4\cos^3(\alpha) - 3\cos(\alpha))(2\cos^2(\alpha) - 1) \\
& - (3\sin(\alpha) - 4\sin^3(\alpha))2\sin(\alpha)\cos(\alpha) \\
=& 8\cos^5(\alpha) - 4\cos^3(\alpha) - 6\cos^3(\alpha) + 3\cos(\alpha) \\
& - 6\sin^2(\alpha)\cos(\alpha) + 8\sin^4(\alpha)\cos(\alpha) \\
=& 8\cos^5(\alpha) - 4\cos^3(\alpha) - 6\cos^3(\alpha) + 3\cos(\alpha) \\
& - 6(1 - \cos^2(\alpha))\cos(\alpha) + 8(1 - \cos^2(\alpha))^2\cos(\alpha) \\
=& 8\cos^5(\alpha) - 10\cos^3(\alpha) + 3\cos(\alpha) \\
& - 6\cos(\alpha) + 6\cos^3(\alpha) + 8\cos(\alpha) - 16\cos^3(\alpha) + 8\cos^5(\alpha) \\
=& (8 + 8)\cos^5(\alpha) + (-10 + 6 - 16)\cos^3(\alpha) + (3 - 6 + 8)\cos(\alpha) \\
\cos(5\alpha) =& 16\cos^5(\alpha) - 20\cos^3(\alpha) + 5\cos(\alpha) \qquad (10)
\end{aligned}$$

Calculamos $\sin(5\alpha)$ usando (3), con 2α y 3α:

$$
\begin{aligned}
\sin(5\alpha) =& \sin(3\alpha)\cos(2\alpha) + \cos(3\alpha)\sin(2\alpha) \\
=& (3\sin(\alpha) - 4\sin^3(\alpha))(1 - 2\sin^2(\alpha)) \\
& + (4\cos^3(\alpha) - 3\cos(\alpha))2\sin(\alpha)\cos(\alpha) \\
=& 3\sin(\alpha) - 6\sin^3(\alpha) - 4\sin^3(\alpha) + 8\sin^5(\alpha) \\
& + 8\cos^4(\alpha)\sin(\alpha) - 6\sin(\alpha)\cos^2(\alpha) \\
=& 3\sin(\alpha) - 10\sin^3(\alpha) + 8\sin^5(\alpha) \\
& + 8(1 - \sin^2(\alpha))^2\sin(\alpha) - 6\sin(\alpha)(1 - \sin^2(\alpha)) \\
=& 3\sin(\alpha) - 10\sin^3(\alpha) + 8\sin^5(\alpha) \\
& + 8(1 - 2\sin^2(\alpha) + sin^4(\alpha))\sin(\alpha) - 6\sin(\alpha)(1 - \sin^2(\alpha)) \\
=& 3\sin(\alpha) - 10\sin^3(\alpha) + 8\sin^5(\alpha) \\
& + 8\sin(\alpha) - 16\sin^3(\alpha) + 8\sin^5(\alpha) - 6\sin(\alpha) + 6\sin^3(\alpha) \\
=& (3 + 8 - 6)\sin(\alpha) + (-10 - 16 + 6)\sin^3(\alpha) + (8 + 8)\sin^5(\alpha) \\
\sin(5\alpha) =& 5\sin(\alpha) - 20\sin^3(\alpha) + 16\sin^5(\alpha) \qquad (11)
\end{aligned}
$$

Es admirable la similitud de las identidades (11) y (10).

$$
\begin{aligned}
\cos(5\alpha) =& 5\cos(\alpha) - 20\cos^3(\alpha) + 16\cos^5(\alpha) \\
\sin(5\alpha) =& 5\sin(\alpha) - 20\sin^3(\alpha) + 16\sin^5(\alpha)
\end{aligned}
$$

Vamos a calcular:

$$\cos\left(\frac{2\pi}{5}\right) \text{ y } \sin\left(\frac{2\pi}{5}\right)$$

Tomemos $\alpha = \frac{2\pi}{5}$ en (11) y (10), obtenemos:

$$\cos(2\pi) = 5\cos\left(\frac{2\pi}{5}\right) - 20\cos^3\left(\frac{2\pi}{5}\right) + 16\cos^5\left(\frac{2\pi}{5}\right) \qquad (12)$$

$$\sin(2\pi) = 5\sin\left(\frac{2\pi}{5}\right) - 20\sin^3\left(\frac{2\pi}{5}\right) + 16\sin^5\left(\frac{2\pi}{5}\right) \qquad (13)$$

Además, tenemos que añadir una relación más:

$$1 = \cos^2\left(\frac{2\pi}{5}\right) + \sin^2\left(\frac{2\pi}{5}\right)$$

Y si sustituimos:

$$
\begin{aligned}
\cos(2\pi) = 1, \quad & c := 5\cos\left(\frac{2\pi}{5}\right), \\
\sin(2\pi) = 0, \quad & s := 5\sin\left(\frac{2\pi}{5}\right)
\end{aligned}
$$

Obtenemos:

$$1 = c^2 + s^2 \tag{14}$$
$$1 = 5c - 20c^3 + 16c^5 \tag{15}$$
$$0 = 5s - 20s^3 + 16s^5 \tag{16}$$

De (16) y (14):

$$0 = s(5 - 20s^2 + 16s^4) \tag{17}$$
$$\tag{18}$$

Tenemos la primera solución del sistema

$$s_1 = 0$$
$$c_1 = 1$$

que no es consistente con $c = \cos(\frac{2\pi}{5})$ ni con $s = \sin(\frac{2\pi}{5})$.

Como c_1 es solución de (15), podemos notar que:

$$5c - 20c^3 + 16c^5 = 1$$
$$5c - 20c^3 + 16c^5 - 1 = 0$$

y por división sintética entre $c - 1$:

$$(c-1)(16c^4 + 16c^3 - 4c^2 - 4c + 1) = 0$$

Consideremos entonces el polinomio $f(c) = 16c^4+16c^3-4c^2-4c+1$, hagamos una depresión de manera que cancelemos el término de tercer grado.

$$f(c) = 16c^4 + 16c^3 - 4c^2 - 4c + 1$$
$$f'(c) = 64c^3 + 48c^2 - 8c - 4$$
$$f''(c) = 192c^2 + 96c^- 8$$
$$f'''(c) = 384c + 96$$
$$f''''(c) = 384$$

Obtengamos el valor de c donde la tercera derivada se hace cero:

$$384c + 96 = 0$$
$$c = \frac{-96}{384}$$
$$c = -\frac{1}{4}$$

Tenemos entonces:

$$
\begin{aligned}
f\left(-\frac{1}{4}\right) &= 16\left(-\frac{1}{4}\right)^4 + 16\left(-\frac{1}{4}\right)^3 - 4\left(-\frac{1}{4}\right)^2 - 4\left(-\frac{1}{4}\right) + 1 \\
&= \frac{1}{16} - \frac{1}{4} - \frac{1}{4} + 1 + 1 = \frac{1 - 4 - 4 + 32}{16} = \frac{25}{16} \\
f'\left(-\frac{1}{4}\right) &= 64\left(-\frac{1}{4}\right)^3 + 48\left(-\frac{1}{4}\right)^2 - 8\left(-\frac{1}{4}\right) - 4 \\
&= -1 + 3 - 2 - 4 = 0 \\
f''\left(-\frac{1}{4}\right) &= 192\left(-\frac{1}{4}\right)^2 + 96\left(-\frac{1}{4}\right)^- 8 \\
&= 12 - 24 + 8 = -20 \\
f'''\left(-\frac{1}{4}\right) &= 384\left(-\frac{1}{4}\right) + 96 \\
&= -96 + 96 = 0 \\
f''''\left(-\frac{1}{4}\right) &= 384
\end{aligned}
$$

Ahora desarrollemos el polinomio $f(c)$ alrededor de $c = \frac{1}{4}$:

$$
\begin{aligned}
f(c) = {} & \frac{f(c)}{0!}\left(c + \frac{1}{4}\right)^0 + \frac{f'(c)}{1!}\left(c + \frac{1}{4}\right)^1 + \frac{f''(c)}{2!}\left(c + \frac{1}{4}\right)^2 \\
& + \frac{f'''(c)}{3!}\left(c + \frac{1}{4}\right)^3 + \frac{f''''(c)}{4!}\left(c + \frac{1}{4}\right)^4 \\
& = \frac{\frac{25}{16}}{1} + \frac{0}{1}\left(c + \frac{1}{4}\right)^1 + \frac{-20}{2}\left(c + \frac{1}{4}\right)^2 \\
& + \frac{0}{6}\left(c + \frac{1}{4}\right)^3 + \frac{384}{24}\left(c + \frac{1}{4}\right)^4 \\
& = 16\left(c + \frac{1}{4}\right)^4 - 10\left(c + \frac{1}{4}\right)^2 + \frac{25}{16}
\end{aligned}
$$

Este polinomio podría factorizarse fácilmente con el objetivo de encontrar sus raíces:

$$
\begin{aligned}
f(c) &= \left[4\left(c+\frac{1}{4}\right)^2\right]^2 - 10\left(c+\frac{1}{4}\right)^2 + \left(\frac{5}{4}\right)^2 \\
&= \left[4\left(c+\frac{1}{4}\right)^2\right]^2 - 2\left[4\left(c+\frac{1}{4}\right)^2\right]\frac{5}{4} + \left(\frac{5}{4}\right)^2 \\
&= \left[4\left(c+\frac{1}{4}\right)^2 - \frac{5}{4}\right]^2 = \left[4\left(c^2+2c\frac{1}{4}+\frac{1}{16}\right) - \frac{5}{4}\right]^2 \\
&= \left(4c^2+2c+\frac{1}{4}-\frac{5}{4}\right)^2 = \left(4c^2+2c-1\right)^2
\end{aligned}
$$

Para factorizar este polinomio, encontraremos la raíces de la ecuaciòn $4c^2 + 2c - 1 = 0$:

$$
\begin{aligned}
c &= \frac{-(2) \pm \sqrt{((2)^2 - 4(4)(-1)}}{2(4)} \\
&= \frac{-2 \pm \sqrt{(4+16)}}{2(4)} \\
&= \frac{-2 \pm \sqrt{20}}{2(4)} \\
&= \frac{-2 \pm \ 2\sqrt{5}}{2(4)} \\
&= \frac{-1 \pm \ \sqrt{5}}{4}
\end{aligned}
$$

Tenemos entonces:

$$
\begin{aligned}
16c^5 - 20c^3 + 5c - 1 &= (c-1)\left(4c^2+2c-1\right)^2 \\
&= (c-1)\left(c - \frac{\sqrt{5}-1}{4}\right)^2 \left(c - \frac{-\sqrt{5}-1}{4}\right)^2
\end{aligned}
$$

Como $c = \cos\left(\frac{2\pi}{5}\right)$, y como ese ángulo está en el primer cuadrante, debemos tener que:

$$
\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}
$$

y, usando la ecuación (14), debemos tener que:

$$\begin{aligned}
s^2 &= 1 - c^2 \\
&= 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2 \\
&= 1 - \frac{5 - 2\sqrt{5} + 1}{16} \\
&= 1 - \frac{6 - 2\sqrt{5}}{16} \\
&= 1 + \frac{-6 + 2\sqrt{5}}{16} \\
&= \frac{16 - 6 + 2\sqrt{5}}{16} \\
&= \frac{10 + 2\sqrt{5}}{16} \\
&= \frac{5 + \sqrt{5}}{8} \\
s &= \pm\sqrt{\frac{5 + \sqrt{5}}{8}}
\end{aligned}$$

De nuevo, como $s = \sin\left(\frac{2\pi}{5}\right)$ y como ese ángulo está en el primer cuadrante, debemos tener:

$$\sin\left(\frac{2\pi}{5}\right) = \sqrt{\frac{5 + \sqrt{5}}{8}}$$

2. Cálculo de $\cos\left(\frac{\pi}{15}\right)$

Sabemos que:

$$\frac{\pi}{15} = \frac{2\pi}{5} - \frac{\pi}{3}$$

y que:

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

por lo tanto:

$$\begin{aligned}
\cos\left(\frac{\pi}{15}\right) &= \cos\left(\frac{2\pi}{5} - \frac{\pi}{3}\right) \\
&= \cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{3}\right) \\
&= \frac{\sqrt{5}-1}{4}\left(\frac{1}{2}\right) + \sqrt{\frac{5+\sqrt{5}}{8}}\left(\frac{\sqrt{3}}{2}\right) \\
&= \frac{\sqrt{5}-1}{8} + \frac{\sqrt{3}}{4}\sqrt{\frac{5+\sqrt{5}}{2}}
\end{aligned}$$

3. Cálculo de $\cos\left(\frac{2\pi}{15}\right)$

Sabemos que:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Tomando $\beta = \alpha$:

$$\cos(2\alpha) = \cos(\alpha)^2 - \sin(\alpha)^2$$

pero

$$\sin(\alpha)^2 = 1 - \cos(\alpha)^2$$

por lo tanto:

$$\cos(2\alpha) = 2\cos(\alpha)^2 - 1$$

tomemos $\alpha = \frac{\pi}{15}$, primero calculamos $cos(\alpha)^2$:

$$
\begin{aligned}
\cos\left(\frac{\pi}{15}\right)^2 &= \left(\frac{\sqrt{5}-1}{8} + \frac{\sqrt{3}}{4}\sqrt{\frac{5+\sqrt{5}}{2}}\right)^2 \\
&= \frac{5-2\sqrt{5}+1}{64} + \frac{3}{16}\left(\frac{5+\sqrt{5}}{2}\right) + 2\left(\frac{\sqrt{5}-1}{8}\right)\frac{\sqrt{3}}{4}\sqrt{\frac{5+\sqrt{5}}{2}} \\
&= \frac{6-2\sqrt{5}}{64} + \frac{15+3\sqrt{5}}{32} + \frac{\sqrt{3}}{16}\sqrt{\frac{(\sqrt{5}-1)^2(5+\sqrt{5})}{2}} \\
&= \frac{3-\sqrt{5}}{32} + \frac{15+3\sqrt{5}}{32} + \frac{\sqrt{3}}{16}\sqrt{\frac{(6-2\sqrt{5})(5+\sqrt{5})}{2}} \\
&= \frac{18+2\sqrt{5}}{32} + \frac{\sqrt{3}}{16}\sqrt{\frac{(30+6\sqrt{5}-10\sqrt{5}-10)}{2}} \\
&= \frac{9+\sqrt{5}}{16} + \frac{\sqrt{3}}{16}\sqrt{\frac{(20-4\sqrt{5}}{2}} \\
&= \frac{9+\sqrt{5}}{16} + \frac{\sqrt{3}}{16}\sqrt{10-2\sqrt{5}} \\
&= \frac{9+\sqrt{5}}{16} + \frac{1}{16}\sqrt{30-6\sqrt{5}}
\end{aligned}
$$

De donde:

$$
\begin{aligned}
\cos\left(2\frac{\pi}{15}\right) &= 2\cos\left(\frac{\pi}{15}\right)^2 - 1 \\
&= \frac{9+\sqrt{5}}{8} + \frac{1}{8}\sqrt{30-6\sqrt{5}} - 1 \\
\cos\left(\frac{2\pi}{15}\right) &= \frac{1+\sqrt{5}}{8} + \frac{1}{8}\sqrt{30-6\sqrt{5}}
\end{aligned}
$$

4. Sustitución $x = 2\cos\left(\frac{12}{15}\right)$

Por último consideremos:

$$
\begin{aligned}
x &= 2\cos\left(\frac{12}{15}\right) \\
&= 2\left(\frac{1+\sqrt{5}}{8} + \frac{1}{8}\sqrt{30-6\sqrt{5}}\right) \\
&= \frac{1}{4}\left(1+\sqrt{5}+\sqrt{30-6\sqrt{5}}\right)
\end{aligned}
$$

Para de simplifar la notación, llamaremos $u = \sqrt{30-6\sqrt{5}}$, y recordaremos que $u^2 = 30 - 6\sqrt{5}$, entonces:

$$
\begin{aligned}
x &= \frac{1}{4}\left(1+\sqrt{5}+u\right) && (19)\\
x^2 &= \frac{1}{16}\left(\left(1+\sqrt{5}\right)^2 + 2\left(1+\sqrt{5}\right)u + u^2\right)\\
&= \frac{1}{16}\left(6+2\sqrt{5}+2\left(1+\sqrt{5}\right)u + 30 - 6\sqrt{5}\right)\\
&= \frac{1}{16}\left(36-4\sqrt{5}+2\left(1+\sqrt{5}\right)u\right)\\
&= \frac{1}{8}\left(18-2\sqrt{5}+\left(1+\sqrt{5}\right)u\right) && (20)\\
x^3 &= x(x^2)\\
&= \frac{1}{32}\left(1+\sqrt{5}+u\right)\left(18-2\sqrt{5}+\left(1+\sqrt{5}\right)u\right)\\
&= \frac{1}{32}\left[\left(1+\sqrt{5}\right)\left(18-2\sqrt{5}\right)+\right.\\
&\quad \left.\left(\left(1+\sqrt{5}\right)^2+\left(18-2\sqrt{5}\right)\right)u + \left(1+\sqrt{5}\right)u^2\right]\\
&= \frac{1}{32}\left[\left(8+16\sqrt{5}\right)+(24)\,u+\left(1+\sqrt{5}\right)\left(30-6\sqrt{(5)}\right)\right]\\
&= \frac{1}{32}\left[\left(8+16\sqrt{5}\right)+24u+24\sqrt{5}\right]\\
&= \frac{1}{4}\left(1+5\sqrt{5}+3u\right) && (21)\\
x^4 &= \left[\frac{1}{8}\left(18-2\sqrt{5}+\left(1+\sqrt{5}\right)u\right)\right]^2\\
&= \frac{1}{64}\left[\left(18-2\sqrt{5}\right)^2+2\left(18-2\sqrt{5}\right)\left(1+\sqrt{5}\right)u+\left(1+\sqrt{5}\right)^2u^2\right]\\
&= \frac{1}{64}\left[\left(344-72\sqrt{5}\right)+2\left(8+16\sqrt{5}\right)u+\left(6+2\sqrt{5}\right)\left(30-6\sqrt{5}\right)\right]\\
&= \frac{1}{64}\left[\left(344-72\sqrt{5}\right)+\left(16+32\sqrt{5}\right)u+\left(120+24\sqrt{5}\right)\right]\\
&= \frac{1}{64}\left[464-48\sqrt{5}+4\left(4+8\sqrt{5}\right)u\right]\\
&= \frac{1}{16}\left[116-12\sqrt{5}+\left(4+8\sqrt{5}\right)u\right] && (22)
\end{aligned}
$$

Vamos a reunir a todas las potencias de x y vamos a prepararlas para hacer la suma:

$$\begin{aligned}
x^4 &= \frac{1}{16}\left[116 - 12\sqrt{5} + \left(4 + 8\sqrt{5}\right)u\right] \\
x^3 &= \frac{1}{4}\left(1 + 5\sqrt{5} + 3u\right) \\
x^2 &= \frac{1}{8}\left(18 - 2\sqrt{5} + \left(1 + \sqrt{5}\right)u\right) \\
x &= \frac{1}{4}\left(1 + \sqrt{5} + u\right) \\
1 &= 1
\end{aligned}$$

Preparamos denominadores:

$$\begin{aligned}
x^4 &= \frac{1}{16}\left[116 - 12\sqrt{5} + \left(4 + 8\sqrt{5}\right)u\right] \\
x^3 &= \frac{1}{16}\left(4 + 20\sqrt{5} + 12u\right) \\
x^2 &= \frac{1}{64}\left(144 - 16\sqrt{5} + \left(8 + 8\sqrt{5}\right)u\right) \\
x &= \frac{1}{64}\left(16 + 16\sqrt{5} + 16u\right) \\
1 &= \frac{1}{16}\left(1\right)
\end{aligned}$$

Preparamos coefientes:

$$\begin{aligned}
x^4 &= \frac{1}{16}\left[116 - 12\sqrt{5} + \left(4 + 8\sqrt{5}\right)u\right] \\
-x^3 &= -\frac{1}{16}\left(4 + 20\sqrt{5} + 12u\right) \\
-4x^2 &= -\frac{1}{16}\left(144 - 16\sqrt{5} + \left(8 + 8\sqrt{5}\right)u\right) \\
4x &= \frac{1}{16}\left(16 + 16\sqrt{5} + 16u\right) \\
1 &= \frac{1}{16}\left(1\right)
\end{aligned}$$

Suma

$$\begin{aligned}
x^4 - x^3 - 4x^2 + 4x + 1 &= \frac{1}{16}\left[(116 - 4 - 144 + 16 + 16)\right. \\
&\quad + \left(-12 - 20 + 16 + 16\right)\sqrt{5} \\
&\quad \left.+ (4 + 8\sqrt{5} - 12 - 8 - 8\sqrt{5} + 16)u\right] \\
&= \frac{1}{16}(0 + 0\sqrt{5} + 0u) \\
&= 0
\end{aligned}$$

como quiería probarse.