Encontrar la suma:

$$\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} \cdots \frac{2n+1}{n^2(n+1)^2}$$

Solución Primero notemos que:

$$\frac{1}{k^2(k+1)^2} = \frac{1}{k^2} - \frac{1}{(k+1)}$$

En efecto:

$$\begin{aligned}
\frac{1}{k^2} - \frac{1}{(k+1)^2} &= \frac{(k+1)^2 - k^2}{k^2(k+1)^2} \\
&= \frac{(k^2+2k+1) - k^2}{k^2(k+1)^2} \\
&= \frac{2k+1}{k^2(k+1)^2}
\end{aligned}$$

Ahora, podemos encontrar la suma:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} &= \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{(k+1)} \\
&= \sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{(k+1)} \\
&= \sum_{k=1}^{n} \frac{1}{k^2} - \sum_{i=2}^{n} +1\frac{1}{i} \\
&= \sum_{k=1}^{n} \frac{1}{k^2} - \sum_{i=2}^{n} +1\frac{1}{i} \\
&= \sum_{k=1}^{n} \frac{1}{k^2} - \sum_{i=2}^{n} +1\frac{1}{i} \\
&= \frac{1}{1^2} + \sum_{k=1}^{n} \frac{1}{k^2} - \sum_{i=2}^{n} \frac{1}{i^2} - \frac{1}{(n+1)^2}
\end{aligned}$$

En la expresión anterior, las sumas son la misma

$$
\begin{aligned}
&= 1 - \frac{1}{(n+1)^2} \\
&= \frac{(n+1)^2 - 1}{(n+1)^2} \\
&= \frac{(n^2 + 2n + 1 - 1}{(n+1)^2} \\
&= \frac{(n^2 + 2n)}{(n+1)^2} \\
\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} &= \frac{n(n+2)}{(n+1)^2}
\end{aligned}
$$